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New variable separation approach: application to nonlinear diffusion equations

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Abstract
The concept of the derivative-dependent functional separable solution (DDFSS), as a generalization to the functional separable solution, is proposed. As an application, it is used to discuss the generalized nonlinear diffusion equations based on the generalized conditional symmetry approach. As a consequence, a complete list of canonical forms for such equations which admit the DDFSS is obtained and some exact solutions to the resulting equations are described.

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1. Introduction

A number of methods has been proved to be effective for finding symmetry reduction and constructing exact solutions to nonlinear diffusion equations. These include the Lie's classical approach [1], the nonclassical approach [2], the direct method [3], the modified direct method [4], the generalized conditional symmetry (GCS) method [5], the nonlocal symmetry method [6], the truncated Painlevé approach [7], the sign-invariant and invariant space methods [8], the transformation method [9] and the ansatz-based methods [10–16]. There are many different directions of the mathematical and physical theory to concern their exact solutions and various properties.

It is well known that the method of variable separation is one of the most universal and efficient means for study of linear partial differential equations (PDEs). Several methods of variable separation for nonlinear PDEs such as the classical method [16], the differential Stäckel matrix approach [17], the ansatz-based method [12–18], the geometrical method [19], the formal variable separation approach (nonlinearization of the Lax pairs or symmetry constraints) [20] and the multilinear variable separation approach [21] have been suggested. From the point of view of symmetry group and the ansatz of the solution form, we now emphasize two of those ansatze. One is the ordinary additive or product separable solution. The other is the functional separable solution (FSS) which is a generalization of the former, where the compatibility of the symmetry constraint with the considered equations is concerned [11–13]. In [21], exact solutions depending on arbitrary functions and their derivatives to many (2+1)-dimensional nonlinear integrable models have emerged through the variable separation process, and abundant localized excitations and their rich interaction behaviour has been revealed. All these prompt us to extend our results in [13, 14] to be more general.

In [12–14], the authors had discussed the FSS

$$f(u) = a(x) + b(t) \tag{1.1}$$

to the generalized porous medium equation

$$u_t = \left(D(u)u_x^n\right)_x + F(u) \tag{1.2}$$

for $n = 1$ and $n \neq 1$ by using the GCS method. In [15], we had investigated the FSS of the more generalized nonlinear diffusion equation

$$u_t = A(u, u_x)u_{xx} + B(u, u_x). \tag{1.3}$$

To obtain more abundant exact solutions to nonlinear PDEs, our basic idea is to weaken the general symmetry constraint condition and to include many more solutions. In this paper, we extend the concept of FSS (1.1) to that of derivative-dependent functional separable solution (DDFSS)

$$f(u, u_x) = a(x) + b(t) \tag{1.4}$$

and apply it to the generalized nonlinear diffusion equation (1.3). It is clear that when $f_{u_x}(u, u_x) = 0$, (1.4) becomes (1.1), so we assume that $f_{u_x}(u, u_x) \neq 0$ hereafter.

The compatibility of (1.4) and (1.3) can be described in terms of the GCS method. The GCS is a natural generalization of both the generalized symmetry and the conditional symmetry [5].

Consider the general *m*th order (1+1)-dimensional evolution equation

$$u_t = E(t, x, u, u_1, u_2, \ldots, u_m) \tag{1.5}$$

where $u_k = \frac{\partial^k u}{\partial x^k}$, $1 \leqslant k \leqslant m$, and E is a smooth function of the indicated variables. Let

$$V = \eta(t, x, u, u_1, u_2, \ldots, u_j)\frac{\partial}{\partial u} \tag{1.6}$$

be an evolutionary vector field and η its characteristic.

Definition 1. *The evolutionary vector field (1.6) is said to be a generalized symmetry of (1.5) if and only if*

$$V^{(m)}(u_t - E)|_L = 0$$

where L is the solution set of (1.5), and $V^{(m)}$ is the mth prolongation of V.

Definition 2. *The evolutionary vector field (1.6) is said to be a GCS of (1.5) if and only if*

$$V^{(m)}(u_t - E)|_{L\cap W} = 0 \tag{1.7}$$

where W is the set of equations $D_x^i\eta = 0$, $i = 0, 1, 2, \ldots$.

It follows from (1.7) that (1.5) admits the GCS (1.6) if and only if

$$D_t\eta = 0 \tag{1.8}$$

where D_t denotes the total derivative in t. Moreover, if η does not depend on time t explicitly, then

$$\eta' E|_{L\cap W} = 0$$

where

$$\eta'(u)E = \lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\eta(u+\epsilon E)$$

denotes the Fréchet derivative of η along the direction E.

The outline of this paper is as follows. In section 2, we will classify equation (1.3) which admits DDFSS. Some exact solutions to the resulting equations are presented in section 3. Section 4 presents a summary and discussion.

2. Equations with DDFSSs

In [13], we have proved the following theorem:

Theorem 1. *Equation (1.5) possesses the additive separable solution*

$$u = a(x) + b(t)$$

if and only if it admits the GCS

$$V = u_{xt}\frac{\partial}{\partial u}. \tag{2.1}$$

Theorem 2. *Equation (1.5) possesses the DDFSS (1.4) if and only if it admits the GCS*

$$V = \frac{\left(f_{uu}u_t + f_{uu_x}u_{xt}\right)u_x + \left(f_{uu_x}u_t + f_{u_xu_x}u_{xt}\right)u_{xx} + f_{u_x}u_{xxt} + f_u u_{xt}}{f_u}\frac{\partial}{\partial u}. \tag{2.2}$$

Proof. Let $v = f(u, u_x) = a(x) + b(t)$, in theorem 1, after replacing u by $v = f(u, u_x)$, and simplifying (2.1), we get

$$v = f(u, u_x) = a(x) + b(t)$$

if and only if

$$V = v_{xt}\frac{\partial}{\partial v} = \frac{\left[\left(f_{uu}u_t + f_{uu_x}u_{xt}\right)u_x + \left(f_{uu_x}u_t + f_{u_xu_x}u_{xt}\right)u_{xx} + f_{u_x}u_{xxt} + f_u u_{xt}\right]}{f_u}\frac{\partial}{\partial u}. \tag{2.3}$$

And then the assertion holds. □

From theorem 2, we know that equation (1.3) admits the DDFSSs (1.4) if and only if it admits the GCS (2.3).

By using the Leibnitz rule on nth order differentiation of product functions, we arrive at the following lemma:

Lemma 1. *Assume $G(r) \neq 0$, $F(r)$ and $G(r)$ are arbitrary smooth functions, then $D_r^i F(r) = 0$, $i = 0, 1, \ldots, N$, if and only if $D_r^i\left(\frac{F(r)}{G(r)}\right) = 0$, $i = 0, 1, \ldots, N$.*

In order to cover the special case $f_u = 0$ in (1.4), on the basis of lemma 1, we can take away the denominator f_u in (2.2) and choose η and V as the following form:

$$V = \left[\left(f_{uu}u_t + f_{uu_x}u_{xt}\right)u_x + \left(f_{uu_x}u_t + f_{u_xu_x}u_{xt}\right)u_{xx} + f_{u_x}u_{xxt} + f_u u_{xt}\right]\frac{\partial}{\partial u}. \tag{2.4}$$

The invariant condition for (2.4) reads

$$\begin{aligned}
V^{(2)}&(u_t - A(u,u_x)u_{xx} - B(u,u_x)) \\
&= D_t\eta - \big(A_u\eta + A_{u_x}D_x\eta\big)u_{xx} - A(u,u_x)D_x^2\eta - \big(B_u\eta + B_{u_x}D_x\eta\big) \\
&= D_t\eta = 0
\end{aligned} \tag{2.5}$$

whenever $D_x^i\eta = 0$ ($i = 0, 1, 2, \ldots$) and $u_t = A(u,u_x)u_{xx} + B(u,u_x)$, where

$$\eta \equiv \big(f_{uu}u_t + f_{uu_x}u_{xt}\big)u_x + \big(f_{uu_x}u_t + f_{u_xu_x}u_{xt}\big)u_{xx} + f_{u_x}u_{xxt} + f_u u_{xt}. \tag{2.6}$$

Substituting (1.3) into (2.6) gives

$$\begin{aligned}
\eta = {}& \big(u_{xx}f_{uu_x} + u_x f_{uu}\big)(Au_{xx}+B) + \big(u_{xx}f_{u_xu_x} + u_x f_{uu_x} + f_u\big)(Au_{xx}+B)_x + f_{u_x}(Au_{xx}+B)_{xx} \\
= {}& f_{u_x}Au_{xxxx} + \big(f_{u_x}A_{u_xu_x} + f_{u_xu_x}A_{u_x}\big)u_{xx}^3 + \big(\big(f_{uu_x}A_{u_x} + f_{u_xu_x}A_u + 2f_{u_x}A_{uu_x}\big)u_x \\
& + f_{u_x}A_u + f_{uu_x}A + f_uA_{u_x} + f_{u_x}B_{u_xu_x} + f_{u_xu_x}B_{u_x}\big)u_{xx}^2 + \big(\big(f_{uu_x}B_{u_x} + 2f_{u_x}B_{uu_x} \\
& + f_uA_u + f_{uu}A + f_{u_xu_x}B_u\big)u_x + \big(f_{u_x}A_{uu} + f_{uu_x}A_u\big)u_x^2 \\
& + \big(3f_{u_x}A_{u_x} + f_{u_xu_x}A\big)u_{xxx} + f_{u_x}B_u + f_uB_{u_x} + f_{uu_x}B\big)u_{xx} \\
& + \big(f_uA + f_{u_x}B_{u_x} + \big(f_{uu_x}A + 2f_{u_x}A_u\big)u_x\big)u_{xxx} \\
& + \big(f_{uu_x}B_u + f_{u_x}B_{uu}\big)u_x^2 + (f_{uu}B + f_uB_u)u_x = 0.
\end{aligned} \tag{2.7}$$

In order to determine all the possible f, A and B from (2.5), a straightforward substitution leads to

$$\begin{aligned}
D_t\eta = \frac{\partial}{\partial t}\big[& f_{u_x}Au_{xxxx} + \big(f_{u_x}A_{u_x}\big)_{u_x}u_{xx}^3 + \big(\big(f_{uu_x}A_{u_x} + f_{u_xu_x}A_u + 2f_{u_x}A_{uu_x}\big)u_x \\
& + f_{u_x}A_u + f_{uu_x}A + f_uA_{u_x} + f_{u_x}B_{u_xu_x} + f_{u_xu_x}B_{u_x}\big)u_{xx}^2 + \big(\big(f_{uu_x}B_{u_x} + 2f_{u_x}B_{uu_x} \\
& + f_uA_u + f_{uu}A + f_{u_xu_x}B_u\big)u_x + \big(f_{u_x}A_{uu} + f_{uu_x}A_u\big)u_x^2 \\
& + \big(3f_{u_x}A_{u_x} + f_{u_xu_x}A\big)u_{xxx} + f_{u_x}B_u + f_uB_{u_x} + f_{uu_x}B\big)u_{xx} \\
& + \big(f_uA + f_{u_x}B_{u_x} + \big(f_{uu_x}A + 2f_{u_x}A_u\big)u_x\big)u_{xxx} \\
& + \big(f_{uu_x}B_u + f_{u_x}B_{uu}\big)u_x^2 + (f_{uu}B + f_uB_u)u_x\big].
\end{aligned} \tag{2.8}$$

Using the integrable condition between (1.3) and $\eta = 0$, we can express $u_{xxxxxxx}$, u_{xxxxxx} and u_{xxxx} in terms of u, u_x, u_{xx} and u_{xxx}, and substituting these expressions into (2.8), we have

$$\begin{aligned}
D_t\eta = {}& (h_1u_{xx} + h_2)u_{xxx}^2 + \big(h_3u_{xx}^3 + h_4u_{xx}^2 + h_5u_{xx} + h_6\big)u_{xxx} \\
& + h_7u_{xx}^5 + h_8u_{xx}^4 + h_9u_{xx}^3 + h_{10}u_{xx}^2 + h_{11}u_{xx} + h_{12} = 0
\end{aligned} \tag{2.9}$$

or equivalently

$$h_i = h_i(u,u_x) = 0 \qquad i = 1, 2, \ldots, 12 \tag{2.10}$$

where the expressions for h_i are complicated, and are given in the appendix. Equation (1.3) possesses the DDFSSs (1.4) if only (2.9) holds, or equivalently, the system of PDEs (2.10) holds. From the system (2.10), one obtains the following relations among A, B and f:

$$\begin{aligned}
0 = {}& (3 + 5\sqrt{1+g_0(u)})\ln A - \sqrt{1+g_0(u)}\ln\big[-3{A_{u_x}}^2g_1(u)(2+g_0(u)) - g_2(u) \\
& - 2g_0(u)A^3 + 2\sqrt{3{A_{u_x}}^2g_1(u)(1+g_0(u)) + 2\sqrt{3}g_0(u)A^3}\sqrt{g_1(u)}A_{u_x}\big] \\
& - 2\ln\big[A_{u_x}\sqrt{3}\sqrt{g_1(u)(1+g_0(u))} + \sqrt{3{A_{u_x}}^2g_1(u)(1+g_0(u)) + 2g_0(u)A^3}\big]
\end{aligned} \tag{2.11}$$

$$f_{u_x} = f_0(u)\frac{A_{u_x}}{A}\exp\left(-\frac{2}{3g_1(u)}\int\frac{A^2}{A_{u_x}}\,\mathrm{d}u_x\right) \tag{2.12}$$

$$B = \int \left\{ \int \left[\left(-4 f_{u_x u_x} f_{uu_x} u_x A^2 - 2 f_{u_x u_x} f_u A^2 - 2\left(f_{u_x}\right)^2 A_u A + 2\left(f_{u_x}\right)^2 A_u A_{u_x} u_x \right.\right.\right.$$
$$+ 3 f_{u_x} A f_u A_{u_x} - 2\left(f_{u_x}\right)^2 A u_x A_{uu_x} + 3 f_{u_x} f_{uu_x} A u_x A_{u_x} + 2 f_{u_x} f_{uu_x} A^2$$
$$\left.\left.\left. + 2 u_x f_{uu_x u_x} f_{u_x} A^2\right) \frac{1}{A^2 f_{u_x}^2} \right] \mathrm{d}u_x + b_0(u) \right\} A\, \mathrm{d}u_x + b_1(u) \tag{2.13}$$

where $f_0(u)$, $g_0(u)$, $g_1(u)$, $g_2(u)$, $b_0(u)$ and $b_1(u)$ are arbitrary functions of u.

It seems that it is impossible to obtain the general solution $A(u, u_x)$ from the transcendental equation (2.11) for arbitrary $g_0(u)$, $g_1(u)$ and $g_2(u)$. It is clear that in order to find an explicit solution of $A(u, u_x)$ from (2.11), the only possible cases are: (i) the factor A_{u_x} appears only in one of the two logarithmic functions and (ii) the ratio of the coefficients of the two logarithmic functions of (2.11) is integer. After a lengthy computation and tedious analysis, we finally attain the following results:

Theorem 3. *The equation*

$$u_t = A(u, u_x) u_{xx} + B(u, u_x)$$

admits nontrivial DDFSSs of form (1.4) with $f_{u_x}(u, u_x) \neq 0$, *if it is locally equivalent to one of the following equations, up to equivalence under translation and dilatation of u:*

(1)

$$u_t = \exp[c_3\phi + \phi_u u_x] \left[u_{xx} + \frac{\phi_{uu}}{\phi_u} u_x^2 + c_3 u_x + c_1 \phi_u^{-1} \right] + c_2 \phi_u^{-1} \tag{2.14}$$

$$f(u, u_x) = \phi_u u_x + c_3\phi + c_4 \tag{2.15}$$

(2)

$$u_t = u_{xx} + c_1 u_x + \frac{\phi_{uu}}{\phi_u} u_x^2 + (c_4 + c_5\phi)\phi_u^{-1} \tag{2.16}$$

$$f(u, u_x) = \phi_u u_x + c_3\phi + c_2 \tag{2.17}$$

(3)

$$u_t = (c_1 u + c_2)(-u_x)^{\alpha-1} u_{xx} - \frac{2c_2}{1+\alpha}(-u_x)^{\alpha+1} + c_4 u + c_3 \tag{2.18}$$

$$f(u, u_x) = \ln(-u_x) \tag{2.19}$$

(4)

$$u_t = (-u_x)^{\alpha-1} u_{xx} + c_2(-u_x)^{\alpha} + c_3 u + c_4 \tag{2.20}$$

$$f(u, u_x) = \ln(-u_x) \tag{2.21}$$

(5)

$$u_t = u_x u_{xx} + c_3 {u_x}^2 + c_4 u^2 + c_2 u + c_1 \tag{2.22}$$

$$f(u, u_x) = \ln(-u_x) \tag{2.23}$$

(6)

$$u_t = \frac{1}{c_3(c_1u+c_2)-u_x}\left[(c_1u+c_2)u_{xx} - 2c_1u_x^2 + \left(\left(2c_3c_1^2 - \frac{c_1c_4}{c_2}\right)u + 2c_3c_1c_2 - c_4\right)u_x \right.$$
$$\left. + c_3c_1\left(c_3c_1^2 + \frac{c_1c_4}{c_2}\right)u^2 + 2c_1c_3(c_4 - c_3c_1c_2)u + c_2c_3(c_4 - c_3c_1c_2)\right] \tag{2.24}$$

$$f(u, u_x) = \ln[c_3(c_1u+c_2) - u_x] \tag{2.25}$$

(7)

$$u_t = (c_1u + c_3 - u_x)^{-1}\big[u_{xx} - (c_4u + c_2)u_x + c_1c_4u^2$$
$$+ \big(c_1c_2 - {c_1}^2 + c_3c_4\big)u + c_3(c_2 - c_1)\big] \tag{2.26}$$

$$f(u, u_x) = \ln(c_1u + c_3 - u_x) \tag{2.27}$$

(8)

$$u_t = (c_1 - u_x)^{-1}u_{xx} - c_3\ln(c_1 - u_x) + c_4u + c_2 \tag{2.28}$$

$$f(u, u_x) = \ln(c_1 - u_x) \tag{2.29}$$

(9)

$$u_t = [c_1(c_2u + c_3) - u_x]^2\big[u_{xx} + (c_2u + c_3)u_x^2 - \big(2c_1c_2^2u^2 + 4c_1c_2c_3u + c_4 + 2c_1{c_3}^2\big)u_x$$
$$+ {c_1}^2c_2^3u^3 + 2{c_1}^2c_2^2c_3u^2 + c_1c_2\big(c_4 + 3c_1c_3^2 - c_1c_2\big)u$$
$$+ c_1^2c_3^3 + c_1c_3c_4 - c_1^2c_2c_3\big] \tag{2.30}$$

$$f(u, u_x) = \ln[c_1(c_2u + c_3) - u_x] \tag{2.31}$$

(10)

$$u_t = (c_1 - u_x)^{\alpha}u_{xx} + c_2 \qquad \alpha \neq -1 \quad \alpha \neq -2 \quad c_1 \neq 0 \tag{2.32}$$

$$f(u, u_x) = \ln(c_1 - u_x) \tag{2.33}$$

(11)

$$u_t = u_{xx} + c_4 \tag{2.34}$$

$$f(u, u_x) = c_2\,\mathrm{arcsinh}[\tan(u_x + c_1)] + c_3 \tag{2.35}$$

(12)

$$u_t = \frac{6c_3^2}{(\phi_uu_x - c_3\phi - c_3c_4)^2}\left[-u_{xx} - \frac{\phi_{uu}}{\phi_u}u_x^2 + 2c_3u_x - c_3^2\phi\phi_u^{-1} - c_3^2c_4\phi_u^{-1}\right] \tag{2.36}$$

$$f(u, u_x) = \frac{1}{(\phi_uu_x - c_3\phi - c_3c_4)^2}\big[c_1\phi_u^2u_x^2 - 2c_1c_3(\phi + c_4)\phi_uu_x + c_1c_3^2(\phi + c_4)^2 + c_2c_3^2\big] \tag{2.37}$$

(13)

$$u_t = \frac{6\phi^2}{(u_x - c_3\phi)^2}\left[-u_{xx} + \frac{6\phi_u + c_4}{6\phi}u_x^2 - \frac{c_3c_4}{3}u_x + \frac{1}{6}c_3^2c_4\phi\right] \tag{2.38}$$

$$f(u, u_x) = \frac{c_1{u_x}^2 - 2c_1c_3\phi u_x + \big(c_2 + c_1c_3^2\big)\phi^2}{(u_x - c_3\phi)^2} \tag{2.39}$$

(14)

$$u_t = \frac{-3(c_4\phi - c_3)^2}{2(\phi u_x - 1)^2}\left[u_{xx} + \frac{\phi_{uu}}{\phi_u}u_x^2\right] \tag{2.40}$$

$$f(u, u_x) = \frac{1}{4(\phi_u u_x - 1)^2}\left[4c_1\phi_u^2 u_x^2 - 8c_1\phi_u u_x + c_2c_4^2\phi^2 - 2c_2c_3c_4\phi + 4c_1 + c_2c_3^2\right] \tag{2.41}$$

(15)

$$u_t = -\frac{-6\phi^2}{(u_x - c_4\phi)^2 - \phi^2}\left[u_{xx} - \left(\frac{\phi_u}{\phi} + c_3\right)u_x^2 + 2c_3c_4u_x + c_3\left(1 - c_4^2\right)\phi\right] \tag{2.42}$$

$$f(u, u_x) = c_1 \ln\left[\frac{(u_x - c_4\phi)^2}{(u_x - c_4\phi)^2 - \phi^2}\right] + c_2 \tag{2.43}$$

(16)

$$u_t = \frac{\sqrt{6}}{3}g_u u_x\left[u_{xx} + \frac{g_{uu}}{g}u_x^3 + \frac{\sqrt{6}}{2}c_1u_x + \frac{c_5}{3c_2g_u}(g + c_4)\right] \tag{2.44}$$

$$f(u, u_x) = c_2 \ln\left[\frac{g_u u_x}{\sqrt{c_3}}\right] \tag{2.45}$$

(17)

$$u_t = \frac{\sqrt{6}}{3}u_x\phi_u u_x u_{xx} + \frac{\sqrt{6}}{18(c_2\phi - 2c_3)}\left[6(c_2\phi - 2c_3)\phi_{uu} - 5c_2\phi_u^2\right]u_x^3 \tag{2.46}$$

$$f(u, u_x) = c_1 \ln\left(\sqrt{\frac{3c_1}{c_2\phi - 2c_3}}\phi_u u_x\right) \tag{2.47}$$

(18)

$$\begin{aligned} u_t = {} & \left(\sqrt{\frac{2}{3}}\phi u_x + c_2\right)u_{xx} + \sqrt{\frac{2}{3}}\phi_u u_x^3 + \frac{c_2}{c_5}\left(c_4\phi - \frac{(4c_3 - 3c_5)\phi_u}{\phi}\right)u_x^2 \\ & + \frac{\sqrt{6}c_2^2}{c_5}\left(c_4 - \frac{2(2c_3 - c_5)\phi_u}{\phi^2}\right)u_x - \frac{3c_2^3(2c_3 - c_5)\phi_u}{c_5\phi^3} \\ & + \frac{c_2\left(3c_4c_2^2 + 2c_1c_5\right)}{2c_5}\phi^{-1} \end{aligned} \tag{2.48}$$

$$f(u, u_x) = \ln(\sqrt{6}\phi u_x + 3c_2) \tag{2.49}$$

where $\phi(u)$ and $g(u)$ are arbitrary functions of u, and α_i, μ, c_i, $i = 1, 2, \ldots,$ are arbitrary constants.

3. Explicit exact DDFSSs

In this section, we deduce exact solutions of the equations obtained in the last section by means of the DDFSS ansatz (1.3). We affirm that the resulting equations enjoy abundant exact solutions due to their inclusive arbitrary functions and constants. Now we just give some of exact solutions result from the derivative-dependent functional separable procedure for all the models listed in theorem 3.

Example 1. For equation (2.14), to obtain exact solutions via derivative-dependent functional separable procedure, one solves the DDFSS ansatz (1.4) with (2.15) first and then substitutes the result into the original equation (2.14) to fix the concrete functions $a(x)$, $b(t)$ and the integration function. Finally, we find that (2.14) has an implicit separable solution

$$\phi(u) = tc_2 + \frac{c_1}{c_3^2} + \frac{\ln(c_2c_3) - \ln(1 - \lambda\exp[c_2c_3(t + a_2)])}{c_3} + a_2c_2 - \frac{c_1x}{c_3} + \left[a_1 - \frac{\exp(c_3x)(c_3x - 1 + \ln(c_3c_1(c_1 - c_3)))}{c_3} - \frac{\mu\ln(\lambda\exp[c_2c_3(t + a_2)] - 1)}{c_3\lambda}\right]\exp(-c_3x)$$

for $c_3 \neq 0$, where and hereafter λ, μ, a_i, c_i, $i \in Z$ are arbitrary constants.

For $c_3 = 0$, equation (2.14) has two DDFSS given implicitly by

$$\phi(u) = \frac{1}{2c_1}\ln\left(\frac{c_1}{1 - \exp(c_1(x + a_1))\lambda}\right)\ln\left[\frac{c_1(1 - \exp(c_1(x + a_1))\lambda)}{\exp(2c_1(x + a_1))\lambda^2}\right] - \frac{1}{2}c_1x^2 - \frac{1}{2}\frac{\left(2a_1c_1^2 - 2a_0c_1 + 2c_1c_4\right)}{c_1}x - (\lambda\ \exp(c_1(-c_4 + a_0)) - c_2)t - \frac{1}{c_1}\left[\mathrm{dilog}\left(\frac{\exp(c_1(x + a_1))\lambda}{\exp(c_1(x + a_1))\lambda - 1}\right) - a_2c_1\right]$$

and

$$\phi(u) = \left(\int \ln\frac{(\mu c_1x - \mu - c_1\lambda)\exp(c_1x) - \exp(a_1)}{c_1^2\exp(c_1x)}\,\mathrm{d}x - x\ln[\mu(t + a_3)] + c_2t + a_2\right)$$

where the function $\mathrm{dilog}(x)$ is the usual dilogarithm function defined by

$$\mathrm{dilog}(x) = \int_1^x \frac{\ln(t)}{1 - t}\,\mathrm{d}x.$$

Example 2. In the same way as for the last example, we can find that equation (2.16) has the implicit separable solution

$$\phi(u) = \frac{1}{-2c_3c_5\left(c_5 + c_3^2 - c_1c_3\right)}\left\{\left[2(c_4c_3\exp(c_3x) - a_1c_5c_3\exp\left(\left(c_5 + c_3^2 - c_1c_3\right)t\right) + a_2c_5\left(c_5 + c_3^2 - c_1c_3\right)\exp(c_3x + c_5t)) - c_3a_4c_5\left(2c_3 - c_1 - \sqrt{c_1^2 - 4c_5}\right) \times \exp\left(\tfrac{1}{2}\left(2c_3 - c_1 + \sqrt{c_1^2 - 4c_5}\right)x\right) - a_3c_3c_5\left(2c_3 - c_1 + \sqrt{c_1^2 - 4c_5}\right) \times \exp\left(-\tfrac{1}{2}\left(c_1 - 2c_3 + \sqrt{c_1^2 - 4c_5}\right)x\right)\right]\exp(-c_3x)\right\} \qquad (3.1)$$

for $c_5 \neq 0$.

If $c_5 = 0$, the equation has the DDFSS

$$\phi(u) = -\frac{c_4 x}{c_1} - \frac{a_1 \exp(-c_1 x)}{c_1(c_3 - c_1)} + \frac{c_3[c_4 + \mu + c_1(\mu t - c_2 + a_2 + a_4)] - \mu}{c_3^2 c_1} + a_3 \exp[-c_3(x + (c_1 - c_3)t)].$$

Note that for $c_1 = 0, c_4 = 0, c_5 = 0, \phi(u) = \mathrm{e}^u$, the equation turns into the potential Burgers equation

$$u_t = u_{xx} + u_x^2.$$

It is related to the usual Burgers equation

$$v_t = v_{xx} + 2vv_x$$

by $v = u_x$.

In this case, the DDFSS (3.1) becomes

$$v = u_x = \left\{\ln\left[(a_4 + a_2 + a_3)c_3^{-1} + a_1 \exp\left(c_3^2 t - c_3 x\right)\right]\right\}_x.$$

Usually, if the inverse function of $\phi(u)$ in (2.16) is well defined then solution (3.1) denotes a multiple soliton resonant solution. For instance, setting

$$\phi(u) = \tan u$$

the nonlinear diffusion equation (2.16) becomes

$$u_t = u_{xx} + c_1 u_x + 2u_x^2 \tan(u) + c_4 \cos^2(u) + \tfrac{1}{2} c_5 \sin(2u) \tag{3.2}$$

and the corresponding solution (3.1) becomes

$$u = \arctan\left\{\frac{2a_3 \exp\left[-\frac{1}{2}\left(\sqrt{c_1^2 - 4c_5} + c_1\right)x\right]}{2c_3 - c_1 - \sqrt{c_1^2 - 4c_5}} + \frac{2a_4 \exp\left[\frac{1}{2}\left(\sqrt{c_1^2 - 4c_5} - c_1\right)x\right]}{2c_3 - c_1 + \sqrt{c_1^2 - 4c_5}} + \frac{a_2}{c_3}\exp(c_5 t) - \frac{c_4}{c_5} + a_1 \exp\left[\left(c_3^2 - c_3 c_1 + c_5\right)t - c_3 x\right]\right\}. \tag{3.3}$$

When $a_2 = a_3 = a_4 = 0, a_1 \neq 0$ solution (3.3) denotes a travelling kink solution. Equation (3.3) is a static kink solution for $a_2 = a_1 = a_4 = 0, a_3 \neq 0$ or $a_2 = a_3 = a_1 = 0, a_4 \neq 0$. Equation (3.3) becomes an instanton solution for $a_1 = a_3 = a_4 = 0, a_2 \neq 0$. Generally, solution (3.3) is a resonant solution of the travelling kink, static kink and the instanton excitations. Figure 1 is the evolution plot of the single kink solution (3.3) with

$$\begin{array}{llll} a_1 = 1 & a_2 = 0 & a_3 = 0 & a_4 = 0 \\ c_4 = 4 & c_5 = 4 & c_1 = 5 & c_3 = 3. \end{array} \tag{3.4}$$

Figure 2 is the evolution plot of the resonant solution of the travelling kink and the static kink while the corresponding parameters are taken as follows:

$$\begin{array}{llll} a_1 = 1 & a_2 = 0 & a_3 = 1 & a_4 = 0 \\ c_4 = 4 & c_5 = 4 & c_1 = 5 & c_3 = 3. \end{array} \tag{3.5}$$

The resonant solution shown in figure 2 denotes the fusion interaction between kink and anti-kink. Before the interaction, there is one large travelling kink and one small travelling kink. After the interaction, the large kink and the small anti-kink degenerate to a single smaller static kink. The soliton fusion and fission phenomena can be found in many (1+1)-dimensional integrable models [22] and have been observed in some real physical systems such as in organic

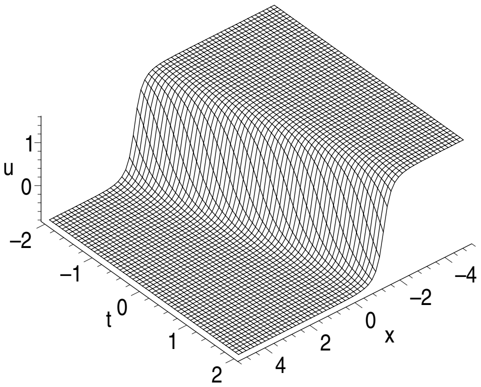

Figure 1. Evolution plot of the single kink solution (3.1) with (3.4).

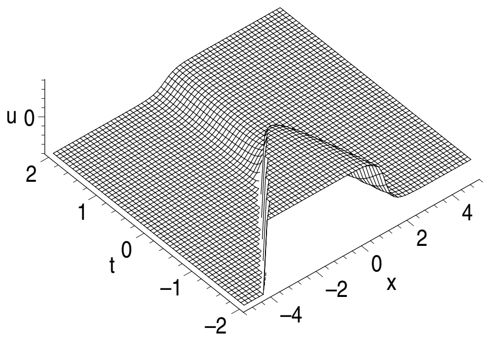

Figure 2. Kink fusion interaction expressed by (3.1) with (3.5).

membranes and macromolecule material [23], in Sr–Ba–Ni oxidation crystals and waveguides [24], in even-clump DNA [25] and in many physical fields such as plasma physics, nuclear physics, hydrodynamics and so on [26].

Example 3. Equation (2.18) possesses an explicit separable solution

$$u = \bigg[\int [-\gamma(c_1c_3 - c_2c_4 + 2(-1)^{1+2\alpha}c_2c_3)\exp(a_2\alpha c_4) + (\alpha-1)c_4t - c_3\exp(-c_4t)]$$
$$\times [(-c_1 + 2c_2(-1)^{2\alpha})\gamma \exp(c_4\alpha(t+a_2)) - 1]$$
$$\times [-c_1 - c_1\alpha + 2\alpha c_2(-1)^{2\alpha}]/[\alpha(-c_1 + 2c_2(-1)^{2\alpha})]\,\mathrm{d}t + a_3\bigg]$$
$$\times [(-c_1 + 2c_2(-1)^{2\alpha})\gamma \exp(c_4\alpha(t+a_2)) - 1]$$
$$\times c_1/[\alpha(-c_1 + 2c_2(-1)^{2\alpha})]\exp(c_4t) - \frac{\gamma^{1/\alpha}(\alpha(x+a_1))^{(1+\alpha)/\alpha}}{1+\alpha}$$
$$\times \left[\frac{c_4\exp(\alpha_4c_4(t+a_2))}{(c_1 + 2(-1)^{1+2\alpha}c_2)\gamma\exp(c_4\alpha(t+a_2)) + 1}\right]^{1/\alpha}.$$

If $c_1 = 0, c_2 = (-1)^n, c_3 = 0, \alpha = n$, the above equation becomes

$$u_t = -u_x{}^{n-1}u_{xx} + \frac{2}{1+n}u_x^{n+1} + c_4u$$

which is just equation (A.2) of theorem 2 in [14] for

$$g_1 = 0 \qquad b_1 = -1 \qquad b_2 = 0 \qquad \beta = -2 \qquad \gamma = -c_4.$$

Here we obtain its new variable separation solution, the DDFSS, which is given explicitly by

$$u = \left\{\int \frac{\gamma c_4 \exp[c_4(a_2n + t(n-1))]}{2\gamma\exp[c_4n(t+a_2)] - (-1)^n}\,\mathrm{d}t + a_3\right\}\exp(c_4t) - \frac{1}{1+n}[n(x+a_1)]^{(1+n)/n}$$
$$\times \left\{\frac{2(-1)^{n+1}\gamma\exp[c_4n(t+a_2)] + 1}{c_4\gamma}\right\}^{1/n}\exp(c_4t + a_2c_4).$$

Example 4. An explicit separable solution of (2.20) reads

$$u = -\left(\left(\int \exp(a(x) + c_3(a_3 + t))\,\mathrm{d}x - \frac{\mu \exp(c_3 t)}{\lambda}\right)\left(\frac{c_3\alpha}{1 - \lambda \exp(c_3(\alpha - 1)(t + a_3))}\right)^{1/(\alpha-1)}\right.$$
$$\left. + a_2 \exp(c_3 t) - \frac{c_4}{c_3}\right.$$

for $c_3 \neq 0$, where $a(x)$ satisfies

$$a''(x) + \alpha a'^2 + c_2 a' - \frac{\lambda}{\alpha}\exp((1 - \alpha)a(x)) = 0.$$

If $c_3 = 0$, an DDFSS is given by

$$u = -\mathrm{e}^{a_0}\int \left(\frac{1}{c_2}\left(a_1 {c_1}^{2/3} \exp\left[\frac{\alpha a_0 {c_1}^{2/3} + \alpha c_2\,\mathrm{e}^{-\alpha a_0} x}{{c_1}^{2/3}}\right] + \alpha a_2 c_2\right)\right)^{1/\alpha}\mathrm{d}x$$
$$+ (c_4 + \alpha\sqrt[3]{c_1}a_2 c_2)t + a_3.$$

Example 5. Equation (2.22) has an explicit separable solution

$$u = -\mathrm{e}^{b(t)}\int \mathrm{e}^{a(x)}\,\mathrm{d}x + s(t)$$

for $c_4 \neq 0$, where $a(x)$, $b(t)$ and $s(t)$ satisfy

$$a''' + (5a'(x) + 2c_3)a'' + 2a'^3 + c_3 a'^2 + 2c_4 = 0$$

$$b(t) = \ln \frac{4a_1\left({c_5}^2 - 4c_1c_4\right)}{4\left({c_5}^2 - 4c_1c_4\right)(\mu^2 + 4\lambda c_4) - {a_1}^2\left(\exp\left(-\sqrt{c_5^2 - 4c_1c_4}\,t\right) + a_2\sqrt{c_5^2 - 4c_1c_4}\right)^2}$$
$$- \sqrt{{c_5}^2 - 2c_1c_4}\,t$$

$$s(t) = \frac{-c_5 + b'(t) + \mu\,\mathrm{e}^{b(t)}}{2c_4}.$$

If $c_4 = 0$, the DDFSS of (2.22) becomes

$$u = \frac{\left(-\lambda^2 c_1 - \mu c_5^2 - c_5^2\lambda \int \exp(a(x))\,\mathrm{d}x\right)\exp(c_5(t+a_0)) + \lambda(-a_1c_5(\exp(c_5 t) + c_1 + \lambda\exp(c_5(2t + a_0))))}{c_5\lambda(\lambda\exp(c_5(t + a_0)) - 1)}$$

where $a(x)$ satisfies

$$a'' - 2a'^2 + c_3 a' - \lambda\,\mathrm{e}^{-a(x)} = 0.$$

Example 6. An explicit variable separable solution of equation (2.24) has the form

$$u = \frac{1}{\mu c_1}\left[2{c_1}^2\exp\left(-\frac{\left(c_1\mu c_2 - 2c_1^2 c_4\right)t + c_2\mu x - 4c_2 a_1 c_1}{2c_1c_2}\right) - c_2\mu\exp(-c_3c_1x)\right.$$
$$\left. + \frac{a_2\mu c_1(\mu c_2 + 2c_1c_4)}{2c_2}t\right]\exp(c_1c_3x).$$

If $c_1 = 0$, the DDFSS of (2.24) is given explicitly by

$$u = -\sqrt{c_2 a_1}(t + a_3)\tan\left(\frac{1}{2}\frac{\sqrt{c_2 a_1}(x + a_2)}{c_2}\right) + c_4 t + c_2 c_3 x + a_4 + \frac{1}{2}a_1 a_2 a_3.$$

If $c_4 = 0$, $c_3 = 0$, equation (2.24) becomes

$$u_t = -\frac{(c_1 u + c_2)u_{xx}}{u_x} + 2c_1 u_x$$

which is equivalent to equation (A.3) of theorem 2 in [14] for

$$g_1 = 0 \qquad \beta = 0 \qquad n = 0 \qquad b_1 = -c_2 \qquad b_2 = -c_1 \qquad g_2 = 0.$$

It has a new solution

$$u = \frac{1}{c_1\mu}\left[2c_1^2 \exp\left(-\frac{c_1\mu t + x\mu - 4a_1c_1}{2c_1}\right) - \mu c_2 + a_2\mu c_1 \exp\left(\frac{1}{2}\mu t\right)\right].$$

Example 7. Equation (2.26) possesses an explicit variable separation solution

$$u = \frac{(\mu c_1 - c_1\exp(c_4(t+a_2)))\int \exp(-c_1x + a(x))\,\mathrm{d}x + a_1c_1c_4\exp(c_4t) - c_3c_4\exp(-c_1x) + \lambda c_1}{c_1c_4\exp(-c_1x)}$$

for $c_1c_4 \neq 0$, where $a(x)$ satisfies

$$a''(x) - c_1a'(x) - \mu\,\mathrm{e}^{a(x)} - c_3c_4 - c_1^2 + c_1c_2 = 0.$$

If $c_1 = 0$, $c_4 = 0$, its DDFSS is given explicitly by

$$u = \frac{(\exp(c_4(t + a_1)) - \lambda)\int \exp(a(x))\,\mathrm{d}x}{c_4} + c_3x + a_2\exp(c_4t) - \frac{c_2 + \mu}{c_4}$$

where $a(x)$ satisfies

$$\pm\int^{a(x)} \frac{1}{\sqrt{-2\lambda\,\mathrm{e}^s + 2c_4c_3s + a_3}}\,\mathrm{d}s = x + a_4.$$

If $c_1 \neq 0$, $c_4 = 0$, its DDFSS is given explicitly by

$$u = \lambda(t + a_1)\exp(c_1x)\int \exp(a(x) - c_1x)\,\mathrm{d}x + (\mu t + a_2)\exp(c_1x) - \frac{c_3}{c_1}$$

where $a(x)$ satisfies

$$a'' - c_1a' + \lambda\,\mathrm{e}^{a(x)} + c_1c_2 - c_1^2 = 0.$$

If both c_1 and c_4 are zero, the DDFSS of (2.26) should be

$$u = \frac{b_1(c_3x + b_2t + b_0)\exp[b_3(x + b_4)] + c_3b_1x + b_1(b_2 - 2b_3c_2)t - 2b_3c_2t_0 + b_0b_1}{b_1[\exp[(x + b_4)b_3] + 1]}$$

with b_0, b_2, b_3, b_4 and t_0 all being arbitrary constants.

Example 8. For $c_4 \neq 0$, the DDFSS of equation (2.28) reads

$$u = \frac{1}{c_4\lambda}(\exp[c_4(t + a_3)] - \lambda)\left[\lambda\int \exp[a(x)]\,\mathrm{d}x - c_3\ln\left(-\frac{\exp[a_3c_4 + c_4t] - \lambda}{c_4}\right)\right] + c_1x + a_2\exp(c_4t) + \frac{c_3\exp(c_4(t + a_3))t}{\lambda} - \frac{\mu + c_2}{c_4}$$

where $a(x)$ satisfies

$$a''(x) + c_3a'(x) + \lambda\,\mathrm{e}^{a(x)} - c_1c_4 = 0.$$

If $c_4 = 0$, the DDFSS of (2.28) is changed to

$$u = -\lambda(t + a_2)\int \mathrm{e}^{a(x)}\,\mathrm{d}x - c_3(t + a_2)\ln[\lambda(t + a_2)] + (\mu + c_3 + c_2)t + c_1x + c_3a_2 + a_1$$

where $a(x)$ satisfies

$$a''(x) + c_3a'(x) - \lambda\,\mathrm{e}^{a(x)} = 0.$$

Example 9.

$$u = \left[-\frac{\sqrt{\exp(2c_2(t+a_2)) - \lambda}(\lambda \int \exp(-c_1c_2x + a(x))\,\mathrm{d}x + \mu)}{\sqrt{c_2}\lambda} + a_1 \exp(c_2 t) \right] \times \exp(c_1c_2x) - \frac{c_3}{c_2}$$

with $a(x)$ being a solution of

$$a''(x) - a'(x)^2 + (c_4 - 2c_1c_2)a'(x) - \lambda\,\mathrm{e}^{2a(x)} + c_1c_2(c_4 - c_1c_2) = 0$$

which is an exact solution of equation (2.30) via the variable separation formula (1.4) with (2.31).

When $c_2 = 0$, the DDFSS is given explicitly by

$$u = -\sqrt{2}\sqrt{\lambda(t+a_2)}\left(\int \mathrm{e}^{a(x)}\,\mathrm{d}x - \mu\lambda\right) + a_1 + c_1c_3x + c_3t$$

where $a(x)$ satisfies

$$a''(x) - a'(x)^2 + c_4a'(x) - \lambda\,\mathrm{e}^{2a(x)} = 0.$$

Example 10. Equation (2.32) has the following special variable separable solution:

$$u = -\left(\frac{1}{\mu\alpha(t+a_4)}\right)^{1/\alpha} \int \mathrm{e}^{a(x)}\,\mathrm{d}x + c_1x - \left(\frac{1}{\mu\alpha(t+a_4)}\right)^{(\alpha+1)/\alpha} \lambda\alpha(t+a_4) + c_2t + a_3$$

where $a(x)$ satisfies

$$\pm\sqrt{(\alpha+2)} \int^{a(x)} \frac{\exp((\alpha+1)\xi)}{\sqrt{a_1 - 2\mu\exp((\alpha+2)\xi)}}\,\mathrm{d}\xi - x - a_2 = 0.$$

Example 11. Equation (2.34) is a trivial linear diffusion equation which allows, of course, infinitely many product variable separation solutions. It is easy to see that it allows a DDFSS which is simply equivalent to a trivial special additive separation solution

$$u = a_1x + c_4t.$$

Example 12. The DDFSS of (2.36) with $c_2 \neq 0$ is given by

$$\phi(u) = -\mathrm{i}\sqrt{c_2}\,\mathrm{e}^x \left[\int \frac{\mathrm{e}^x}{\sqrt{-a_3\,\mathrm{e}^{-2x} - a_2\,\mathrm{e}^{2x} + 2\sqrt{a_2a_3}\tanh(24\sqrt{a_2a_3}(t+a_1)/c_2)}}\,\mathrm{d}x + \frac{1}{a_2a_3}\arctan\left(\sqrt{2\sqrt{a_3a_2}\tanh\left(\frac{48\sqrt{a_2a_3}(t+a_1)}{c_2}\right) - a_3 - a_2}\right) \right] - a_4\,\mathrm{e}^x - c_4.$$

Example 13. Equation (2.38) has the following special solution:

$$\int^u (\phi(s))^{-1}\,\mathrm{d}s = -2\frac{\sqrt{(3\mu^2t + +\sqrt{3}\mu c_2x + 3a_1c_2 + 3a_2c_2 - 3c_1c_2)}}{\mu} + c_3x + c_4t.$$

There are three special cases of (2.38), which are known in the literature:

(i) If $c_3 = 0, \phi(u) = c_0$, (2.38) becomes

$$u_t = -6\frac{u_{xx}}{u_x^2} + c_4 \tag{3.6}$$

which has a DDFSS given explicitly by

$$u = \left(-2\frac{\sqrt{\sqrt{3}\mu c_2 x + 3\mu^2 t + 3(a_1c_2 + a_2c_2 - c_1c_2)}}{\mu} + c_4 t\right).$$

This equation is equivalent to equation (19) of example 3.1 in [14] for $n = -1, \alpha = 0$;

(ii) If $c_3 = 0, \phi(u) = u$, (2.38) is simplified to

$$u_t = -6\frac{u^2 u_{xx}}{u_x^2} + (6 + c_4)u \tag{3.7}$$

which has the DDFSS

$$u = \exp\left(-2\frac{\sqrt{\sqrt{3}\mu c_2 x + 3\mu^2 t + 3(a_1c_2 + a_2c_2 - c_1c_2)}}{\mu} + c_4 t\right).$$

Equation (3.7) is equivalent to equation (26) of example 3.2 in [14] for $n = -1, \alpha = 0$; equation (26) is a generalization of the curve shortening equation

$$w_t = w^2 w_{xx} + w^3.$$

(iii) If $c_4 = 0, c_3 = 0, \phi(u) = \mathrm{e}^u$, (2.38) becomes

$$u_t = -6\frac{\mathrm{e}^{2u} u_{xx}}{{u_x}^2} + 6\,\mathrm{e}^{2u} \tag{3.8}$$

which has the DDFSS

$$u = \ln\left(\frac{\mu}{2\sqrt{\sqrt{3}\mu c_2 x + 3\mu^2 t + 3(a_1c_2 + a_2c_2 - c_1c_2)} - a_3\mu}\right).$$

After transformation $u = w/2$, (3.8) is transformed to

$$w_t = -24\frac{\mathrm{e}^w w_{xx}}{{w_x}^2} + 12\,\mathrm{e}^w$$

which is equivalent to equation (23) in example (3.4) in [14] for $\beta = 0, \sigma = -\frac{1}{2}$, $n = -1, \alpha = 0$.

Example 14. Equation (2.40) admits a special separable solution given implicitly by

$$\phi(u) = -\frac{1}{c_4^2 c_2}\left[2c_4\sqrt{a_0c_2x + 3a_0^2 t - c_1c_2 + b_0c_2} + 2a_0 - c_3c_2c_4 \right.$$
$$\left. - a_1c_4^2c_2 \exp\left(\frac{c_4\left(2\sqrt{a_0c_2x + 3a_0^2t - c_1c_2 + b_0c_2} - 3c_4a_0t\right)}{2a_0}\right)\right].$$

Example 15. The DDFSS of (2.42) has the form

$$\int^u (\phi(s))^{-1}\,\mathrm{d}s = \frac{c_4c_1}{a_2}\pi\,\mathrm{i} + \frac{c_1}{a_2}\ln\left[\exp\left(\frac{3a_2^2t + a_3c_1 + a_1c_1 + c_1c_2}{c_1^2}\right)\right.$$
$$- 2\exp\left(\frac{c_1(2a_3 + 2a_1 + a_2x) + 6a_2^2t}{c_1^2}\right)$$
$$- 2\mathrm{i}\sqrt{-\exp\left(\frac{c_1(a_2x + a_3 + a_1) + 3a_2^2t}{c_1^2}\right) + \exp\left(\frac{c_2}{c_1}\right)}$$
$$\left.\times\exp\left(\frac{c_1(3a_3 + 3a_1 + a_2x) + 9a_2^2t}{2c_1^2}\right)\right] - \frac{(\ln 2)c_1}{a_2}$$
$$+ \frac{c_4a_2c_1x + \left(-3a_2^2 + 6c_3a_2c_1\right)t + \mathrm{i}c_1^2\pi + (a_4a_2 - a_3 - a_1 - c_2)c_1}{c_1a_2}.$$

Example 16.

$$g(u) = \frac{\lambda}{\mu}\exp\left(\frac{2\mu t + 6a_2}{3c_2}\right) + \left[-3c_2a_1\,\exp\left(-\frac{\mu t + 3a_2}{3c_2}\right)\mu^{-1}\right.$$
$$+ \frac{1}{c_1^{3/2}\sqrt{c_2}\mu}\left(-\sqrt{2}\mu\sqrt{-\exp(-\sqrt{6}c_1(x + a_3)) + \lambda} + \sqrt{2}(\ln 2)\mu\sqrt{\lambda}\right.$$
$$+ \sqrt{2}\mu\sqrt{\lambda}\ln\left(\lambda + \sqrt{\lambda}\sqrt{-\exp(-\sqrt{6}c_1(x + a_3)) + \lambda}\right) + \sqrt{3}\mu\sqrt{\lambda}c_1(x + a_3)$$
$$\left.\left. + 3a_4\sqrt{c_3}\mu c_1^{3/2}\sqrt{c_2}\right)\right]\exp\left(\frac{\mu t + 3a_2}{3c_2}\right)$$

is a special DDFSS of (2.44).

Example 17. Equation (2.46) for $c_2 \neq 0$ has a DDFSS determined implicitly by

$$\phi(u) = \frac{1}{2916c_2c_1^3}\left[243c_2^2c_1^2\exp\left(\frac{2a_2}{c_1}\right)\left(\exp\left(\frac{2a_1}{c_1}\right)x^2 + a_3^2\right)\right.$$
$$- 18\sqrt{6}c_2^3c_1\exp\left(\frac{4a_2}{c_1}\right)\left(\exp\left(\frac{4a_1}{c_1}\right)x + a_3\exp\left(\frac{3a_1}{c_1}\right)\right)t$$
$$\left. + 486\exp\left(\frac{2a_2 + a_1}{c_1}\right)c_1^2c_2^2a_3x + 2\exp\left(6\frac{a_2 + a_1}{c_1}\right)c_2^4t^2 + 5832c_1^3c_3\right].$$

For $c_2 = 0$, equation (2.46) enjoys an exact DDFSS in the form

$$\phi(u) = -\frac{\sqrt{6}}{3c_1\lambda^2(t + a_2)}\left(-\mathrm{i}\lambda c_1^{3/2}\sqrt{c_3}\int\exp\left(\frac{a(x)}{c_1}\right)\mathrm{d}x - \mu c_1 + a_1\lambda^2(t + a_2)\right)$$

where $a(x)$ satisfies

$$\sqrt{c_3}\int^{a(x)}\exp\left(\frac{2s}{c_1}\right)\frac{1}{\sqrt{-\exp\left(\frac{3a_3}{c_1}\right) + \mathrm{i}c_1^{3/2}\sqrt{c_3}\lambda\exp\left(\frac{3s}{c_1}\right)}}\,\mathrm{d}s = x + a_4.$$

Example 18. For equation (2.48), a special DDFSS is given by

$$\int^u \phi(s)\,\mathrm{d}s = \frac{1}{\sqrt{6}}\left[\mathrm{e}^{b(t)}\int\mathrm{e}^{a(x)}\,\mathrm{d}x - 3c_2x - s(t)\right]$$

with

$$\frac{\phi_u}{\phi^2} = \frac{1}{6c_2(2c_3 - c_5)\,\mathrm{e}^{2a(x)}\,\mathrm{e}^{2b(t)}}\left[-3\sqrt{6}c_5 b'(t)\int \mathrm{e}^{a(x)}\,\mathrm{d}x\,\mathrm{e}^{b(t)} + 3\sqrt{6}c_5 s'(t) + \sqrt{6}c_5 a'(x)\,\mathrm{e}^{2a(x)}\,\mathrm{e}^{2b(t)} + 3c_4c_2\,\mathrm{e}^{2a(x)}\,\mathrm{e}^{2b(t)} + 18c_1c_2c_5\right].$$

Given any $\phi(u)$, then $a(x)$, $b(t)$ and $s(t)$ are determined by the above two relations, thus the separable solution u is determined.

4. Summary and discussion

In summary, we have brought forward a new conception of DDFSS to nonlinear evolution equations. Taking the generalized nonlinear diffusion equation as a concrete example and using the GCS approach, we have obtained a complete list of explicit canonical forms for such equations which admit the DDFSSs. As a consequence, some exact explicit solutions to the resulting equations have been obtained via solving the DDFSS ansatz (1.4). The approach also provides a symmetry group interpretation to the DDFSSs. Our approach is more general than the others due to the involvement of the derivative-dependent functional separable function in the ansatz (1.4). Subsequently, we can obtain many new nonlinear models which can be solved by means of generalized nonlinear variable separation procedures. Some new exact solutions of some known models are given explicitly. Several different types of localized excitations of some complicated nonlinear diffusion equations have been found via the DDFSS approach.

The GCS approach can be used to find DDFSSs for some other types of nonlinear systems. In [27] and [28], the possible DDFSSs are considered for the Korteweg–de Vries type systems and the nonlinear wave type equations, respectively.

Though the variable separation approach has been developed in several different directions [13–21], it is still far beyond being perfect. There are some important problems that should be studied further. One of the most important problems may be how to unify all the known informal variable separation approaches? Perhaps we can propose a unified variable separation ansatz in a most general way ($u = u(x_1, x_2, \ldots, x_n)$, $G_j \equiv G_j(\xi_1, \ldots, \xi_{m_j})$, $\xi_k = \xi_k(x_1, \ldots, x_n)$, $k = 1, \ldots, m_j, m_j < n$)

$$f(x_1, x_2, \ldots, u, u_{x_i}, u_{x_ix_j}, \ldots) = g(x_1, x_2, \ldots, G_j, G_{j\xi_i}, G_{j\xi_i\xi_k}, \ldots). \qquad (4.1)$$

Though all the ansatze of the known informal variable separation approaches are the special cases of (4.1), the concrete realization procedures for different known approaches are quite different. Can we find a universal method, say, the GCS method, to realize the generalized variable separation ansatz (4.1)?

It is clear that when one writes down the general variable separation ansatz (4.1), one needs to clarify the equivalence problem caused by variable transformations. For instance, the special form of (4.1)

$$f(x_1, x_2, \ldots, u, u_{x_i}, u_{x_ix_j}, \ldots) = g(x_1, x_2, \ldots, x_n, \phi_1(x_1), \phi_2(x_2), \ldots, \phi_n(x_n)) \qquad (4.2)$$

is equivalent to the form

$$F(x_1, x_2, \ldots, u, u_{x_i}, u_{x_ix_j}, \ldots) = G(x_1, x_2, \ldots, x_n, \psi_1(y_1), \psi_2(y_2), \ldots, \psi_n(y_n)) \qquad (4.3)$$

with $y_i = y_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$ being arbitrary functions of $\{x_1, x_2, \ldots, x_n\}$ because we can make the variable transformation $\{x_1, x_2, \ldots, x_n\} \leftrightarrow \{y_1, y_2, \ldots, y_n\}$.

More specifically, the variable separation ansatz (1.3) used in this paper is equivalent to

$$F(u, x_\xi u_x + t_\xi u_t) = G(a(\xi(x, t)) + b(\tau(x, t))) \qquad (4.4)$$

where $\xi = \xi(x, t)$ and $\tau = \tau(x, t)$ are arbitrary functions of $\{x, t\}$, a and b are arbitrary functions of ξ and τ, G is an arbitrary function of $a + b$ while $x = x(\xi, \tau)$ and $t = t(\xi, \tau)$ are determined by the inverse transformations of $\xi = \xi(x, t)$ and $\tau = \tau(x, t)$. The variable separable equations related to the ansatz (4.4) can be simply obtained from theorem 3 by means of the transformations $x \to \xi(x, t)$ and $t \to \tau(x, t)$.

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Appendix. Expressions of h_i of (2.10)

It takes a dozen pages to write down h_i, $i = 1, 2, \ldots, 12$, explicitly in terms of f, A and B. For simplicity, we display them by introducing the following notation:

$$\Gamma_0 = \left(f_{u_x} B_u\right)_u u_x^2 + (f_u B)_u u_x \tag{A.1}$$

$$\Gamma_1 = \left(f_{u_x} A_{u_x}\right)_{u_x} \tag{A.2}$$

$$\Gamma_2 = \left(f_{u_x} A\right)_u + \left[\left(f_{u_x} A_{u_x}\right)_u + \left(f_{u_x} A_u\right)_{u_x}\right] u_x + \left(f_{u_x} B_{u_x}\right)_{u_x} + f_u A_{u_x} \tag{A.3}$$

$$\Gamma_3 = f_{u_x u_x} A + 3 f_{u_x} A_{u_x} \tag{A.4}$$

$$\Gamma_4 = \left(f_{u_x} A_u\right)_u u_x^2 + \left[\left(f_{u_x} B_{u_x}\right)_u + \left(f_{u_x} B_u\right)_{u_x} + (f_u A)_u\right] u_x + (f_u B)_{u_x} + f_{u_x} B_u \tag{A.5}$$

$$\Gamma_5 = \left(2 f_{u_x} A_u + f_{u u_x} A\right) u_x + f_u A + f_{u_x} B_{u_x} \tag{A.6}$$

$$F_i = \Gamma_i f_{u_x}^{-1} A^{-1} \qquad i = 0, 1, \ldots, 5 \tag{A.7}$$

$$G_1 = F_3 F_1 - F_{1u_x} \tag{A.8}$$

$$G_2 = -F_{1u} u_x - F_{2u_x} + F_5 F_1 + F_3 F_2 \tag{A.9}$$

$$G_3 = -3F_1 + F_3^2 - F_{3u_x} \tag{A.10}$$

$$G_4 = F_5 F_2 - F_{2u} u_x + F_3 F_4 - F_{4u_x} \tag{A.11}$$

$$G_5 = -F_{5u_x} + 2F_3 F_5 - F_{3u} u_x - 2F_2 \tag{A.12}$$

$$G_6 = -F_{4u} u_x + F_3 F_0 + F_5 F_4 - F_{0u_x} \tag{A.13}$$

$$G_7 = F_5^2 - F_4 - F_{5u} u_x \tag{A.14}$$

$$G_8 = F_5 F_0 - F_{0u} u_x. \tag{A.15}$$

With the help of the above notation, h_i, $i = 1, 2, \ldots, 12$ in (2.10) read

$$h_1 = -9F_3 A_{u_x} + 2AG_3 + 15A_{u_x u_x} \tag{A.16}$$

$$h_2 = \left(-2F_3 A_u - F_{3u} A + 12A_{u u_x}\right) u_x - 7F_5 A_{u_x} + 4A_u + 3B_{u_x u_x} + F_{5u_x} A + AG_5 \tag{A.17}$$

$$h_3 = \left(5G_3 + F_{3u_x} - 4F_3^2 - F_1\right) A_{u_x} - 3F_3 A_{u_x u_x} + 10A_{u_x u_x u_x} + \left(F_{1u_x} + 4G_1 + G_{3u_x} - 2F_1 F_3\right) A \tag{A.18}$$

$$h_4 = AG_{5u_x} + F_{5u_x}A_{u_x} - 4F_2A_{u_x} - F_3^2B_{u_x} - 2F_3A_u + 3F_1B_{u_x} - 4F_5A_{u_xu_x} + 4A_uu_xG_3 + AG_{3u}u_x + F_{3u_x}A_uu_x + F_{2u_x}A - 3F_1AF_5 - 8F_3A_{u_x}F_5 + B_{u_x}G_3 - 5F_3u_xA_{uu_x} - F_2AF_3 + 5A_{u_x}G_5 + 6F_1A_uu_x - 3F_3^2A_uu_x + 24u_xA_{uu_xu_x} + 3AG_2 + F_{3u}A + F_{3u_x}B_{u_x} + 22A_{uu_x} + 6B_{u_xu_xu_x} \tag{A.19}$$

$$h_5 = \left(18A_{uuu_x} - 2F_3A_{uu}\right)u_x^2 + \left(\left(B_{uu_x} - 6F_5A_u\right)F_3 - 7F_5A_{uu_x} + 4A_uG_5 + F_{3u_x}B_u + AG_{5u} + F_{5u_x}A_u + 12B_{uu_xu_x} + 4F_2A_u + 16A_{uu}\right)u_x + \left(-2F_5B_{u_x} + B_u\right)F_3 - 7F_4A_{u_x} - 4F_5^2A_{u_x} + \left(-2F_2A - 3A_u - B_{u_xu_x}\right)F_5 + F_{3u}B + 10B_{uu_x} + B_uG_5 + F_{5u_x}B_{u_x} + F_{5u}A + AG_{7u_x} + 5A_{u_x}G_7 + 2AG_4 + 2F_2B_{u_x} + AF_{4u_x} \tag{A.20}$$

$$h_6 = 4A_{uuu}u_x^3 + \left(F_3B_{uu} - 3F_5A_{uu} + 6B_{uuu_x}\right)u_x^2 + \left(4A_uG_7 + F_{5u_x}B_u - F_5B_{uu_x} + AG_{7u} + 4B_{uu} + \left(2F_4 - 3F_5^2\right)A_u\right)u_x + \left(B_{u_x} - F_5A\right)F_4 - 10A_{u_x}F_0 + \left(F_{0u_x} + F_0F_3 + G_6\right)A + B_{u_x}\left(G_7 - F_5^2\right) + F_{5u}B \tag{A.21}$$

$$h_7 = A_{u_xu_xu_xu_x} + A\left(G_{1u_x} - 3F_1^2\right) + \left(-7A_{u_xu_x} - AG_3 - 4F_3A_{u_x}\right)F_1 + \left(A_{u_xu_xu_x} + AG_1\right)F_3 + A_{u_x}\left(F_{1u_x} + 5G_1\right) \tag{A.22}$$

$$h_8 = \left[\left(-3F_3A_u - 10A_{uu_x}\right)F_1 + 4A_{uu_xu_xu_x} + AG_{1u} + 3F_3A_{uu_xu_x} + F_{1u_x}A_u + 4A_uG_1\right]u_x + \left(-4A_u - 5F_2A - B_{u_xu_x} - F_3B_{u_x} - 4F_5A_{u_x}\right)F_1 + \left(-4F_3A_{u_x} - 8A_{u_xu_x} - AG_3\right)F_2 + \left(3A_{uu_x} + B_{u_xu_xu_x} + AG_2\right)F_3 + \left(A_{u_xu_xu_x} + AG_1\right)F_5 + B_{u_xu_xu_xu_x} + 5A_{u_x}G_2 - AG_5F_1 + 6A_{uu_xu_x} + AG_{2u_x} + F_{1u_x}B_{u_x} + B_{u_x}G_1 + F_{2u_x}A_{u_x} + F_{1u}A \tag{A.23}$$

$$h_9 = \left(-3F_1A_{uu} + 3F_3A_{uuu_x} + 6A_{uuu_xu_x}\right)u_x^2 + \left[\left(2B_{uu_x} - 3F_5A_u\right)F_1 - 3\left(F_3A_u + 4A_{uu_x}\right)F_2 + \left(3A_{uu} + 3B_{uu_xu_x}\right)F_3 + F_{1u_x}B_u + AG_{2u} + 12A_{uuu_x} + F_{2u_x}A_u + 3F_5A_{uu_xu_x} + 4A_uG_2 + 4B_{uu_xu_xu_x}\right]u_x - AG_5F_2 + \left(2B_u - AG_7 - F_5B_{u_x} - 4F_4A\right)F_1 - 2F_2^2A - \left(F_3B_{u_x} + 2B_{u_xu_x} + 4F_5A_{u_x} + 5A_u\right)F_2 + \left(AG_4 + 3B_{uu_x} - 4F_4A_{u_x}\right)F_3 - \left(9A_{u_xu_x} + AG_3\right)F_4 + \left(3A_{uu_x} + B_{u_xu_xu_x} + AG_2\right)F_5 + 5G_4A_{u_x} + AF_{2u} + F_{4u_x}A_{u_x} + B_{u_x}G_2 + 3A_{uu} + F_{1u}B + 6B_{uu_xu_x} + F_{2u_x}B_{u_x} + AG_{4u_x} \tag{A.24}$$

$$h_{10} = \left(5A_{u_x} + F_3A\right)G_6 + \left(4A_{uuuu_x} + F_3A_{uuu}\right)u_x^3 + \left(3F_1B_{uu} + 3F_5A_{uuu_x} + 6A_{uuu} + 3F_3B_{uuu_x} - 4F_2A_{uu} + 6B_{uuu_xu_x}\right)u_x^2 + \left[3F_5A_{uu} - \left(14A_{uu_x} + 3F_3A_u\right)F_4 - 3F_5A_uF_2 + AG_{4u} + 3F_5B_{uu_xu_x} + 3F_3B_{uu} + 4A_uG_4 + 12B_{uuu_x} + F_{2u_x}B_u + F_{4u_x}A_u\right]u_x - \left(3F_2A + 6A_u + AG_5 + F_3B_{u_x} + 3B_{u_xu_x} + 4F_5A_{u_x}\right)F_4 - \left(10A_{u_xu_x} + 3F_1A + AG_3 + 4F_3A_{u_x}\right)F_0 + 3F_5B_{uu_x} + F_{4u}A + F_2B_u + F_{4u_x}B_{u_x} + B_{u_x}G_4 - AG_7F_2 + 3B_{uu} + F_5AG_4 + AG_{6u_x} + F_{0u_x}A_{u_x} + F_{2u}B - F_5B_{u_x}F_2 \tag{A.25}$$

$$h_{11} = \left(F_5A + 4A_uu_x + B_{u_x}\right)G_6 + \left(5A_{u_x} + F_3A\right)G_8 + A_{uuuu}u_x^4 + \left(F_5A_{uuu} + 4B_{uuuu_x} + F_3B_{uuu}\right)u_x^3 + \left(6B_{uuu} - 5F_4A_{uu} + 2F_2B_{uu} + 3F_5B_{uuu_x}\right)u_x^2 + \left[F_{4u_x}B_u + AG_{6u} + 3F_5B_{uu} + F_{0u_x}A_u - \left(2B_{uu_x} + 3F_5A_u\right)F_4 - \left(3F_3A_u + 16A_{uu_x}\right)F_0\right]u_x - F_4^2A - \left(F_5B_{u_x} + AG_7\right)F_4 - \left(4B_{u_xu_x} + A(G_5 + 2F_2) + F_3B_{u_x} + 4F_5A_{u_x} + 7A_u\right)F_0 + (F_{0u} + G_{8u_x})A + F_{0u_x}B_{u_x} + F_{4u}B \tag{A.26}$$

$$h_{12} = \big(F_5A + 4A_u u_x + B_{u_x}\big)G_8 + B_{uuuu}u_x^4 + F_5u_x^3B_{uuu} - (6A_{uu}F_0 - F_4B_{uu})u_x^2 + \big(AG_{8u} - 4B_{uu_x}F_0 - 3F_5A_uF_0 + F_{0u_x}B_u\big)u_x - F_4AF_0 - F_5B_{u_x}F_0 + (-AG_7 - B_u)F_0 + F_{0u}B. \quad \text{(A.27)}$$

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